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AN ALGORITHM TO MINIMIZE SINGLE VARIABLE POLYNOMIAL FUNCTIONS FROM ANY STARTING POINT WITH QUADRATIC CONVERGENCE

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ABSTRACT

Search methods often display non-convergence or excessive convergence time on certain classes of nonlinear functions arising in engineering design. The authors will define a new geometric -programming based search method for single variable polynomials that displays quadratic convergence from any starting point. Comparison over a group of test problems is made with a version of Newton's method.

AN EXAMPLE OF THE METHOD

The method used in this paper is based on the condensation approach to solving geometric programming problems of Beightler & Phillips, [1], and Passy, [2]. This method will be demonstrated with the following example from Thermopolis and Lehman, [3]. They proposed a model of an inventory situation with a total expected cost given as:

$$Y = K_1 \cdot Q + \frac{K_2}{Q} + K_3 \cdot Q^2 \tag{1}$$

Where the first term is the annual holding cost, the second term in the annual ordering cost and the third term is the annual obsolescence cost, and Q is the order quantity. This type of problem is easily solved with a Newton's method. It is noted, however, that the number of iterations to convergence is somewhat sensitive to starting point using this method. We now propose to define a new method that is absolutely insensitive to starting point as follows using the example problem. We may now start to "condense" this problem by combining terms with positive powers which gives:

$$Y = (K_1 \cdot Q + K_3 \cdot Q^2) + \frac{K_2}{Q} \tag{2}$$

Now following the method of [1] and [2] above, we have that:

$$K_1 \cdot Q + K_3 \cdot Q^2 \geq \left(\frac{K_1 \cdot Q}{d_1}\right)^{(d_1)} \cdot \left(\frac{K_3 \cdot Q^2}{d_1}\right)^{(d_2)}, \tag{3}$$

where $d_1 + d_2 = 1$, and $d_1, d_2 > 0$. (4)

We may now rewrite (3) as:

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$$K_1 \cdot Q + K_3 \cdot Q^2 \geq \left(\left(\frac{K_1}{d_1} \right)^{(d_1)} \cdot \left(\frac{K_3}{d_2} \right)^{(d_2)} \right) \cdot Q^{(d_1 + 2 \cdot d_2)}. \tag{5}$$

Now following the algorithmic approach of Rijckert & Martens, [4], if we choose some starting value for Q say $\overline{Q}$, we may then define d_1 and d_2 as:

$$d_1 = \frac{K_1 \cdot \overline{Q}}{(K_1 \cdot \overline{Q} + K_3 \cdot \overline{Q}^2)}, \text{ and } d_2 = \frac{K_3 \cdot \overline{Q}^2}{(K_1 \cdot \overline{Q} + K_3 \cdot \overline{Q}^2)} \tag{6}$$

It should be noted that the requirement that the d's sum to one, necessary for the inequalities (3) and (5) to be true, is met. If we now define:

$$K_4 = \left(\frac{K_1}{d_1} \right)^{d_1} \cdot \left(\frac{K_3}{d_2} \right)^{d_2} \text{ and } K_5 = d_1 + 2 \cdot d_2 \tag{7}$$

we now have the original problem restated as:

$$TEC = K_4 \cdot Q^{K_5} + \frac{K_2}{Q}. \tag{8}$$

The minimum of this problem using any method is:

$$TEC^* = (K_5 + 1) \cdot \left(\frac{K_4 \cdot K_2^{K_5}}{(K_5^{K_5})} \right)^{\left(\frac{1}{(K_5 + 1)} \right)} \tag{9}$$

This minimum occurs at an optimum order quantity of:

$$Q^* = \left(\frac{K_2}{K_4 \cdot K_5} \right)^{\left(\frac{1}{K_5 + 1} \right)} \tag{10}$$

With the above relationships we may define the flow chart of the algorithm for the solution of this problem on the following page:

FLOWCHART OF THE METHOD FOR THE EXAMPLE PROBLEM

Step 1. Choose a starting value of Q
call it $\overline{Q}$

Step 2. Set
$$d_1 = \frac{K_1 \cdot \overline{Q}}{K_1 \cdot \overline{Q} + K_3 \cdot \overline{Q}^2}, d_2 = \frac{K_3 \cdot \overline{Q}^2}{K_1 \cdot \overline{Q} + K_3 \cdot \overline{Q}^2}$$

Step 3. Set
$$K_4 = \left(\frac{K_1}{d_1}\right)^{d_1} \cdot \left(\frac{K_3}{d_2}\right)^{d_2}, K_5 = d_1 + 2 \cdot d_2$$

Step 4. Set
$$Q_{new} = \left(\frac{K_2}{K_4 \cdot K_5}\right)^{\left(\frac{1}{(K_5+1)}\right)}$$

Step 5. If absolute value of $(Q_{new} - \overline{Q}) < error$

Then Set $\overline{Q} = Q_{new}$

And Go To Step 2.
If not then Go To Step 6.

Step 6. Print optimal $Q^* = Q_{new}$

Let us now put some numbers to the model of Thomopoulus & Lehman giving:

$$TEC = 10 \cdot Q + \frac{4000}{Q} + 10 \cdot Q^2, \qquad (10a)$$

This problem has a minimum at Q= 5.686 which yields a optimum TEC = 1083.648. We now compare the iterations to convergence using this method and using a plain-vanilla Newton's method. By plain vanilla, we mean a Newtons' method that has no adjustment for step size. It is well known with Newton's method that too large a step size will often step over the optimum and too small a step size will not converge within a reasonable time or will get caught in a local optimum. In any case the authors are prepared to provide listings of our Newton's method programs for the examples of this paper in Basic for those interested in confirming our results for themselves. Both the Newton's method and the G. P. method used for all problems in this paper use an allowable error of .001 for stopping criterion. The comparisons are shown below:

Starting Point	N. Iterations	G. P. Iterations
.000001	infeasible	4
.01	18	4
.1	13	4
1	7	4
5.686	0	0
275	25	4
100000	20	4

It is immediately noticeable that the G.P. method converges in the same number of iterations from all of the above starting points. We assert, without proof that the G.P. method proposed in this paper will converge in the same number of iterations over a range of starting points bounded only by the floating point word length of the computer used. The authors are unaware of any other search method that displays this quality at this point in time. In short, this method appears to demonstrate *Quadratic Convergence*. Quadratic Convergence may be defined as convergence to an optimal solution in which the absolute error decreases proportionally to the square of the iteration. Stated another way,if the absolute error for each iteration is plotted against the iterations, the plot would display the general shape of a specific quadratic function described by Freund, [5], as:

$$f(i) = \frac{a^2}{b \cdot i^2}.$$

This,in fact proved to be the situation for all of the problems tested in this paper.

THE GENERAL METHOD (POSITIVE COEFFICIENT TERMS)

Based on the above discussion we now assert that any problem of the form:

$$\text{Min}.Y = \sum_{i=1}^{N} K_i \cdot Q^{A_i} + \sum_{i=1}^{M} C_i \cdot Q^{-B_i} \tag{12}$$

$$Where\, K_i, A_i, C_i, B_i \geq 0$$

may be solved , using condensation, as follows. Again, using Rijckaert & Martens,[4], we note that:

$$\sum_{i=1}^{N} K_i \cdot Q^{A_i} \geq \left(\left(\frac{K_1}{d_1}\right)^{d_1} \cdot \left(\frac{K_2}{d_2}\right)^{d_2} \cdot \ldots \cdot \left(\frac{K_N}{d_N}\right)^{d_N} \right) \cdot Q^{\left(\sum_{i=1}^{N} d_i \cdot A_i\right)} \tag{13}$$

Where, as before, the d's must sum to 1. We now accomplish the same process for the negatively powered terms,which gives:

$$\sum_{i=1}^{M} C_i \cdot Q^{-B_i} \geq \left(\left(\frac{C_1}{e_1}\right)^{e_1} \cdot \left(\frac{C_2}{e_2}\right)^{e_2} \cdot \ldots \cdot \left(\frac{C_M}{e_M}\right)^{e_M} \right) \cdot Q^{-\sum_{i=1}^{M} e_i \cdot B_i} \tag{14}$$

Again, requiring that the e's sum to 1. Now, consistent with the example problem seen before, we define the d's and the e's as function of some starting value of Q, designated as Q bar which gives:

$$d_1 = \frac{K_1 \cdot \overline{Q}^{A_1}}{\sum_{i=1}^{N} K_i \cdot \overline{Q}^{A_i}}, \ldots\ldots, d_N = \frac{K_N \cdot \overline{Q}^{A_N}}{\sum_{i=1}^{N} K_i \cdot \overline{Q}^{A_i}}, \tag{15}$$

$$e_1 = \frac{C_i \cdot \overline{Q}^{-B_1}}{\sum_{i=1}^{M} C_i \cdot \overline{Q}^{-B_i}}, \ldots\ldots, e_M = \frac{C_M \cdot \overline{Q}^{-B_M}}{\sum_{i=1}^{M} C_i \cdot \overline{Q}^{-B_i}}. \tag{16}$$

Now let us define the following quantities:

$$D = \left(\left(\frac{K_1}{d_1}\right)^{d_1} \cdot \left(\frac{K_2}{d_2}\right)^{d_2} \cdot \ldots \cdot \left(\frac{K_N}{d_N}\right)^{d_N} \right), \tag{17}$$

$$E = \sum_{i=1}^{N} d_i \cdot A_i, \tag{18}$$

$$F = \left(\left(\frac{C_1}{e_1}\right)^{e_1} \cdot \left(\frac{C_2}{e_2}\right)^{e_2} \cdot \ldots \cdot \left(\frac{C_M}{e_M}\right)^{e_M} \right), \tag{19}$$

$$G = \sum_{i=1}^{M} e_i \cdot B_i, \tag{20}$$

We now have the original problem of (12), restated in the simple form:

$$Y = D \cdot Q^{E} + F \cdot Q^{-G}. \tag{21}$$

The minimum value of this problem, using any appropriate method is:

$$Y^{*} = (E+G) \cdot \left(\frac{D^{G} \cdot F^{E}}{G^{G} \cdot E^{E}}\right)^{\left(\frac{1}{(E+G)}\right)}, \tag{22}$$

which occurs at an optimum value of Q given by:

$$Q^{*} = \left(\frac{F \cdot G}{D \cdot E}\right)^{\left(\frac{1}{(E+G)}\right)}. \tag{23}$$

With the above definitions we now define the flow chart for the convergent algorithm on the following page.

FLOWCHART OF THE GENERAL METHOD FOR POSITIVE COEFFICIENTS

Step 1. Choose a starting value of Q call it $\overline{Q}$

Step 2. For all i

Set
$$d_i = \frac{K_i \cdot \overline{Q}^{A_i}}{\sum_{i=1}^{N} K_i \cdot \overline{Q}^{A_i}}$$

Set
$$e_i = \frac{C_i \cdot \overline{Q}^{-B_i}}{\sum_{i=1}^{M} K_i \cdot \overline{Q}^{-B_i}}$$

Step 3. Set
$$D = \left(\frac{K_1}{d_1}\right)^{d_1} \cdot \left(\frac{K_2}{d_2}\right)^{d_2} \cdot \ldots \cdot \left(\frac{K_N}{d_N}\right)^{d_N}$$

Set
$$E = \sum_{i=1}^{N} d_i \cdot A_i$$

Set
$$F = \left(\frac{C_i}{e_1}\right)^{e_1} \cdot \left(\frac{C_2}{e_2}\right)^{e_2} \cdot \ldots \cdot \left(\frac{C_M}{e_M}\right)^{e_M}$$

Set
$$G = \sum_{i=1}^{M} d_i \cdot A_i$$

Step 4. Set
$$Q_{new} = \left(\frac{F \cdot G}{D \cdot E}\right)^{\left(\frac{1}{(E+G)}\right)}$$

Step 5. If absolute value of $(Q_{new} - \overline{Q}) < error$

Then set $\overline{Q} = Q_{new}$

And Go To Step 2.
If not, then Go To Step 6.

Step 6. STOP, print optimal $Q^* = Q\,new$

We will exemplify this method using an example of cofferdam design due to Negahabet & Stark [6]. The mathematical description for such a problem is that total cost would be equal to the sums of the excavation and fill costs, piling face cost, piling side cost , and expected flooding cost. The original total cost function was expressed as a function of four design variables, total piling length t, cell face length f, cell side length s, and height above low water h. This model has three constraints in the four design variables. The constraints may be used to eliminate three of the four variables,resulting in a cost function described only in terms of the variable h. This resulting cost function may be written in a specific example due to Wilde [7] as follows:

Minimize: $$TC = 3660 \cdot h + 175 \cdot h^2 + 1.34 \cdot h^3 + 50000 \cdot h^{-1}, \qquad (23a)$$

This cost function has a minimum at h= 3.218, yielding a optimum cost of TC= $29,172.35. We now compare the convergence of this problem between the Newton's method mentioned above and the G. P. method as seen below:

Starting Point	N. Iterations	G. P. Iterations
.000001	infeasible	4
.01	18	4
.1	12	4
1	6	4
3.218	0	0
10	6(wrong answer)	4
1000	13(wrong answer)	4
10000	11	4
275	7	4
100000	16(wrong answer)	4

Again, it is interesting to note that the proposed method's convergence is independent of starting point, while the Newton's method is not. The authors cheerfully admit that all of the problems so far are sufficiently convex to yield global optima. We now proceed to expand the method to problems with multiple optima.

THE GENERAL METHOD (POSITIVE AND NEGATIVE COEFFICIENT TERMS)

Based on the above we now assert that any problem of the form:

$$\text{Min}.Y = \sum_{i=1}^{N} K_i \cdot Q^{A_i} + \sum_{i=1}^{M} C_i \cdot Q^{-B_i} - \sum_{i=1}^{P} H_i \cdot Q^{R_i}, \qquad (24)$$

May be solved using condensation, as follows. We first note that using the general method discussed above with positive coefficient terms , that the above problem may be restated as:

$$\text{Min}.Y = D \cdot Q^{E} + F \cdot Q^{-G} - \sum_{i=1}^{P} H_i \cdot Q^{R_i}. \qquad (25)$$

Problems such as this often display extremely erratic results using Newton based methods, (computational experience will be shown later in this paper). The reason for this erratic performance is that such problems (called signomial geometric programming problems), are often embassingly non-convex and display multiple optima. The authors attempted various approaches to the above problem with little success. Finally, one of the authors, Mr. Greening, suggested that the problem of (25) above could be rewritten as:

Minimize: Y

Subject to: $$Y \geq D \cdot Q^{E} + F \cdot Q^{-G} - \sum_{i=1}^{P} H_i \cdot Q^{R_i}, \tag{26}$$

which was shown to be an *equivalent* problem to (25) by Beightler & Phillips in [1]. Mr. Greening attributes this idea to a paper of Avriel & Williams, [10]. He then suggested simply moving the negative coefficient terms over to the left-hand-side of the problem which gives:

Min: Y

Subject To: $$Y + \sum_{i=1}^{P} H_i \cdot Q^{R_i} \geq D \cdot Q^{E} + F \cdot Q^{-G}. \tag{27}$$

Again following Rijckeart & Martens, [4], we have, as before:

$$Y + \sum_{i=1_i}^{PH} \cdot Q^{R_i} \geq$$

$$\left(\left(\frac{1}{f_1}\right)^{f_1} \cdot \left(\frac{H_1}{f_2}\right)^{f_2} \cdot \left(\frac{H_2}{f_3}\right)^{f_3} \cdot \ldots \cdot \left(\frac{H_P}{f_{P+1}}\right) \cdot Y^{f_1} \cdot Q^{\sum_{i=1}^{P} R_i \cdot f_{i+1}} \right), \tag{28}$$

Where, Y bar is the function value at some starting value of Q, here called Q bar and again, as before, the f's are defined as functions of the same value of Q bar which gives:

$$f_1 = \frac{\overline{Y}}{\overline{Y} + \sum_{i=1}^{P} H Q^{R_i}}, \ldots f_{P+1} = \frac{H_P}{\overline{Y} + \sum_{i=1}^{P} H_i \cdot Q^{R_i}}, \tag{29}$$

If we now define:

$$S = \left(\left(\frac{1}{f_1}\right)^{f_1} \cdot \left(\frac{H_1}{f_2}\right)^{f_2} \cdot \left(\frac{h_2}{f_3}\right)^{f_3} \cdot \ldots \cdot \left(\frac{H_P}{f_{P+1}}\right) \right), \tag{30}$$

and, $T = \sum_{i=1}^{p} R_i \cdot f_{i+1},$ (31)

we have the problem in (27) restated in the simple form given below:

Min Y:

Subject To: $S \cdot Y^{f_1} \cdot Q^T \geq D \cdot Q^E + F \cdot Q^{-G}.$ (32)

Now, by dividing through by the L. H. S. we get the desired reformulation as:

Min Y:

Subject To: $1 \geq \left(\frac{D}{S}\right) \cdot Y^{-f_1} \cdot Q^{(E-T)} + \left(\frac{F}{S}\right) \cdot Y^{-f_1} \cdot Q^{-(G+T)},$ (33)

The minimum value of this problem using any appropriate method is:

$$Y^* = \left(D^G \cdot F^E \cdot \left(\frac{D}{F}\right)^T \cdot \left(\frac{E-T}{G+T}\right)^T \cdot \left(\frac{G+T}{S}\right)\right)^{\left(\frac{1}{f_1}\right)}, \quad (34)$$

which occurs at an optimum value of Q given by:

$$Q^* = \left(\frac{F \cdot (G+T)}{D \cdot (E-T)}\right)^{\left(\frac{1}{E+G}\right)}, \quad (35)$$

With the above relationships we now define the flow chart on the following page.

FLOWCHART OF THE GENERAL METHOD FOR POSITIVE AND NEGATIVE COEFFICIENTS

Step 1. Choose a starting value of Q call it $\overline{Q}$

Step 2. For all i define d's and e's as before

Step 3. Set $\overline{Y} = \sum_{i=1}^{N} K_i \cdot \overline{Q}^{A_i} + \sum_{i=1}^{M} C_i \cdot Q^{-B_i} - \sum_{i=1}^{P} H_i \cdot \overline{Q}^{R_i}$

Step 4. Set
$$f_1 = \frac{\overline{Y}}{\overline{Y} + \sum_{i=1}^{P} H_i \cdot \overline{Q}^{R_i}}$$

Step 5. For all i
Set
$$f_i = \frac{H_i \cdot \overline{Q}^{R_i}}{\overline{Y} + \sum_{i=1}^{P} H_i \cdot \overline{Q}^{R_i}} \, for i = 2,3,\ldots P,$$

Step 6. Set D, E, F, and G as before

Step 7. Set $T = \sum_{i=1}^{P} R_i \cdot f_{i+1}$

Step 8. Set
$$Q_{new} = \left(\frac{F \cdot (G+T)}{D \cdot (E-T)}\right)^{\frac{1}{(E+G)}}$$

Step 9. If absolute value of $(Q_{new} - \overline{Q}) < error$

Then set $\overline{Q} = Q_{new}$

And Go To Step 2.
If not, then Go To Step 10.

Step 10. STOP, print optimal $Q^* = Q_{new}$

Our first example, using this method, is from Schweyer, [8], and concerns minimizing the cost of insulation of a steam piping system as shown below, where S is the percentage of solids between two pieces of equipment:

Minimize: $$TC = 100000 \cdot (S - 0.1)^3 + 2100 \cdot (1 - S) + 1095 \cdot S^{-1} + 875, \tag{36}$$

Performing the required exponentiation and collecting terms this gives:

Minimize: $$TC = 100000 \cdot S^3 - 27900 \cdot S^2 - 1200 \cdot S + 1095 \cdot S^{-1} + 2875, \tag{37}$$

This problem has a global minimum at S= .315, yielding a optimum cost of TC= $6330.40. The comparison of the two methods from various starting points is shown below:

Starting Point	N. Iterations	G. P. Iterations
.000001	infeasible	6
.001	infeasible	6
.01	10	6
.1	4	6
1	3(wrong answer)	6
275	12	6
100000	20	6

A plot of iterations for this problem against absolute error is provided on the next page, which demonstartes the quadratic convergence property of this method on this problem.

Wilde,[7], presented a problem originally formulated by Washington Braga of Brazil. A cubical refrigerated van is to be designed to transport fruit between two cities in Brazil. Information provided includes variables for the number of trips and the length of a side of the van, the thickness of insulation, transportation and labour cost, etc. After performing a few manipulations and substitutions, however, it is found to be possible to write the local expression as a function of the length of the side, S, only as shown below:

Minimize: $$TC = 69.1608 \cdot S^{.5} + 0.0066 \cdot S^2 + 620000000 \cdot S^{-3} - 10 \cdot S^{.5}, \tag{38}$$

This problem has a global minimum at S= 143.6806, yielding an optimum cost of TC= $1054.42. The comparision of this problem on Newton's method and G. P is shown below:

Starting Point	N. Iterations	G. P. Iterations
.000001	infeasible	7
.001	infeasible	7
.01	47	7
.1	36	7
1	26	7
231	27	7
232	infeasible	7
275	infeasible	7
100000	infeasible	7

On these problems it may be seen that Newton's method behaves even more erratically than before, sometimes finding a wrong answer, sometimes not converging within a reasonable limit of iterations. It is also noted that in the cases where N-R did converge that it showed much greater sensitivity to starting points than before. Interestingly enough, it is also seen that the GP method still displays the quadratic convergence from all starting points. It is also significant to note that in every case above, the GP method discovered the global optimum. The authors are quick to state that *no* assertion of global optimality is being made here except for the problems found in this paper. At this point in time convergence to global optimality of this method is unproven and , we suspect, unlikely.

CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY

We have defined, in this paper, a method for minimizing single variable polynomial functions of a certain class. The method displays quadratic convergence from any starting point over an extremely wide range. It is important to note, as in the steam pipe insulation problem of Schweyer , that the GP starting points that are out of the obvious feasible region for the variable, (i.e. between .1 and 1), *have no effect on convergence.* The authors agree immediately that a more sophisticated Newton-Ralphson method or some other search method can be programmed that would yield consistently fewer iterations than the Newton's method used in this paper. We believe, based on our preliminary compuational experience here, that the robustness of the GP method is superior across a wider range of problems than any such particular method could display.

Clearly, a next possible step is to define a method to deal with more general problems of the type:

$$\text{Min.} \quad Y = \sum_{i=1}^{N} K_i \cdot Q^{A_i} + \sum_{i=1}^{M} C_i \cdot Q^{-B_i} - \sum_{i=1}^{P} H_i \cdot Q^{R_i} - \sum_{i=1}^{U} V_i \cdot Q^{-W_i}. \tag{39}$$

This will shortly appear. Another step, also underway, is based on extending the above methods to the multivariable method of condensation of Ratliff,[9].
Preliminary computations on the multivariable engineering design problems attempted to this date also display quadratic convergence.

For our more mathematically inclined readers, we wish to be clear that we have provided a minimal, (if any), foundation mathematically for our approach. We openly admit our collective lack of ability and interest in pursuing the conceptual basis of this method. The challenges to prove under what conditions that this method converges to local or global optima should be a fertile field for those so inclined. We wish those people who might wish to establish, or disestablish, the basis for this method Godspeed and good hunting.

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