ON GAVER'S PARALLEL SYSTEM

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ABSTRACT

Gaver's parallel system (and its variants) has received considerable attention in the literature. The Laplace-transform (LT) of the *survival* function, corresponding to the underlying systems, has been derived by various (alternative) methods. Unfortunately, little-to-no attention has been paid to invert the corresponding transform. First, we present a general reliability analysis of Gaver's basic parallel system (valid for an *arbitrary* repair time distribution). Then, we formulate some general hints to obtain the *numerical* inverse. Finally, we propose a tangible methodology to derive the *exact* inverse in some particular but important cases of *non-rotational* transforms.

OPSOMMING

Gaver se parallelsisteem (en variante daarvan) is reeds dikwels behandel in die literatuur. Die Laplacetransformasie van die *oorlewingsfunksie* van die onderliggende sisteme is reeds afgelei op verskillende uiteenlopende wyses. Ongelukkigerwys is min aandag gegee aan die ooreenstemmende inverse transformasie. Ten aanvang word 'n algemene betroubaarheidsontleding van Gaver se basiese parallelsisteem voorgehou (geldig vir 'n arbitrêre hersteltyd-verdeling). Dit word gevolg deur wenke vir bepaling van die numeriese inverse. Ten slotte word 'n tasbare metodologie vir die afleiding van die presiese inverse vir sekere belangrike nie-rotasionele transformasies voorgestel.

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1. INTRODUCTION

Gaver's parallel system [7,8] (for instance, two generators in *active* redundance [2] attending the light-plant of a tunnel) has received considerable attention in the literature, e.g. [2, 4, 5, 6, 9,10, 11, 12, 13, 14, 15, 16, 18, 19].

The Laplacetransform (LT) of the survival function has been derived by various (alternative) mathematical methods, such as the supplementary variable technique (SVT), e.g. [7, 8, 12] and the phase method, e.g. [11]. The resulting set of transient differential equations are then solved by a LT technique. It should be noted that this particular methodology requires the existence of bounded density functions, hereby imposing tangible (statistical) restrictions on the generality of the survival function. The advanced concept of Lebesgue-Stieltjes integration is rather an exception than a rule. On the other hand, the SVT is not restricted to any particular reliability system. Thus, apart from the restrictions imposed on the underlying distributions, the SVT remains a powerful (but intricate) technique, e.g. [12, 15, 17]. Unfortunately, little-to-no attention had been paid to invert the corresponding LT. As usual, the inversion process is left to the system designer. Consequently, the reliability analysis of Gaver's system (and its variants), subject to arbitrary distributions, is far from complete.

First, we present a general reliability analysis of Gaver's parallel system valid for an *arbitrary* repair time distribution. Then, some general hints are formulated to invert the LT by numerical methods. Finally, a tangible methodology to derive the *exact* inverse in some particular but important cases of *non-rotational* transforms is proposed.

It should be noted that the proposed exact inversion procedure is, in general, quite complicated. Therefore, in order to keep the analysis as simple as possible, we consider the particular but important case of deterministic repair is considered. However, note that the proposed procedure works equally well for any other distribution, provided that certain Cauchy-type integrals (see forthcoming analysis) can be evaluated as a finite sum in terms of linear combinations of known algebraic and/or transcendental functions.

In the opposite case, one is forced to rely upon alternative numerical methods (see forthcoming remarks).

2. FORMULATION

Consider Gaver's parallel system subjected to the usual conditions (i.i.d. random variables, instantaneous switch, perfect repair).

Each operative unit has a constant failure rate $\lambda > 0$ and an arbitrary repair time distribution $R(.), R(0) = 0$. Let $R^{-}(.) \equiv 1 - R(.)$. The repair time is denoted by r.

Let $\{X_t, t \geq 0\}$ be a stochastic process, with arbitrary discrete state space $\{A, B, C\} \subset [0, \infty)$, characterized by the following mutually exclusive events:

$\{X_t = A\}$: "Both units are simultaneously operative at time t."
$\{X_t = B\}$: "One unit is operative and the other unit is under progressive repair at time t."
$\{X_t = C\}$: "Both units are down at time t."

Consider the *stopping* time

$$\Theta := \inf\{t > 0: \quad X_t = C \big| X_0 = B\}.$$

In reliability theory, Θ is usually called the first system-down time. The origin of time is fixed at the instant of the first failure, so that the *busy* period of the repairman starts at $t = 0$, i.e. $X_0 = B$, **P**-a.s. The *survival* function of the system is defined by

$$\Re(t) = \mathbf{P}\{\Theta > t\}, \, t \geq 0.$$

The greatest integer function is denoted by [.].

3. INTEGRAL EQUATION

Observe that state A is *regenerative* for the process $\{X_t, t \geq 0\}$. Hence, by renewal theory,

$$\mathbf{P}\{\Theta < t\} = \int_{x=0}^{t} \lambda e^{-\lambda x} R^{-}(x)dx \quad + \quad \int_{x=0}^{t} \int_{u=0}^{t-x} e^{-\lambda u} dR(u) \int_{v=0}^{x} 2\lambda e^{-2\lambda(x-v)} d\,\mathbf{P}\{\Theta < v\}dx.$$

The Stieltjes-convolution theorem entails that

$$\mathbf{E}e^{-s\Theta} = \frac{\lambda}{s+\lambda} \; \frac{1 - \mathbf{E}e^{-(s+\lambda)r}}{1 - \dfrac{2\lambda}{s+2\lambda}\mathbf{E}e^{-(s+\lambda)r}}, \mathrm{Re}\, s \geq 0. \tag{1}$$

Observe that $\mathbf{E}e^{-s\Theta}$ represents the Laplace-Stieltjes-transform of the distribution of θ, whereas $\mathbf{E}e^{-(s+\lambda)r}$ represents the LT of the exponential tail with respect to the repair time distribution, i.e.

$$\mathbf{E}e^{-(s+\lambda)r} = \int_0^\infty e^{-st} e^{-\lambda t} dR(t)\,.$$

Clearly, Θ is finite **P**-a.s.

4. SURVIVAL FUNCTION

Note that $\Re(.)$ is *uniquely* determined by the Laplace-transform

$$\frac{1 - \mathbf{E}e^{-s\Theta}}{s} = \int_0^\infty e^{-st} \Re(t)dt, \mathrm{Re}\, s \geq 0,$$

where,

$$\left.\frac{1-\mathbf{E}e^{-s\Theta}}{s}\right|_{s=0} := \int_0^\infty \Re(t)dt = \mathbf{E}\Theta < \infty.$$

Remarks

- First consider the case of *Coxian* repair, i.e. let

  $$\mathbf{E}e^{-sr} = \frac{Q_k(s)}{Q_m(s)}, \mathrm{Re}\ s \geq 0$$

  where $Q_n(s);\ n = k, m;\ m > k \geq 0$, is a polynomial of degree n.

  An explicit evaluation of $\Re(.)$ could then be performed by a *numerical* inversion technology, sustained by user-friendly software, such as Wofram [20]. Cf. [17] for further details.

- For a *non-rational* transform (for instance, induced by Weibull repair) an *exact* evaluation of $\Re(.)$ is in general excluded. However, a *numerical* evaluation could then be obtained by an inversion technology developed by Blanc [3].

- For a *particular* family of non-rational transform, the *exact* explicit evaluation of $\Re(.)$ can be achieved by methods of complex analysis. As an application, consider the case of *deterministic* repair.

5. DETERMINISTIC REPAIR

Let

$$R(t) = \begin{cases} 1, & if\ t \geq t_0 > 0, \\ 0, & if\ t < t_{0.} \end{cases}$$

For the sake of simplicity, we take t_0 as time unit. Hence $\mathbf{E}e^{-sr} = e^{-s}$. By (1), we have

$$\mathbf{E}e^{-s\Theta} = \frac{\lambda}{s+\lambda} \frac{1-e^{-(s+\lambda)}}{1-\dfrac{2\lambda}{s+2\lambda}e^{-(s+\lambda)}}.$$

Some algebra entails that

$$\frac{1-\mathbf{E}e^{-s\Theta}}{s} = \frac{1}{s+\lambda}\left(1+\frac{1}{2}\frac{\rho(s)}{1-\rho(s)}\right), \mathrm{Re}\ s \geq 0, \qquad (2)$$

where $\rho(s) := 2\lambda(s+2\lambda)^{-1}e^{-(s+\lambda)}$.

Observe that $\Re(.)$ is continuous on $(0,\infty)$ and of bounded variation on $[0,\infty)$. Hence, by the inversion thereom,

$$\Re(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{-iT}^{iT} e^{st} \frac{1-\mathbf{E}e^{-s\Theta}}{s} ds,\, t>0 .$$

Or, by (2),

$$\Re(t) = e^{-\lambda t} + \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \frac{1}{2} \frac{\rho(s)}{1-\rho(s)} \frac{ds}{s+\lambda} . \tag{3}$$

An application of the maximum modulus theorem [1] entails that

$$|\rho(s)| \le \max_{\mathrm{Re}\, s\ge 0} |\rho(s)| = e^{-\lambda} < 1.$$

Consequently, the identity

$$\frac{\rho(s)}{1-\rho(s)} = \sum_{k=1}^{[t]} \rho(s) + \frac{\rho^{[t]+1}(s)}{1-\rho(s)}$$

holds for Re $s \ge 0$ and $t \ge 0$ (where the empty sum is defined 0). In addition, $\rho(s)$ is analytic in the half-plane $\{s \in C: \ \mathrm{Re}\, s > 0\}$ and boundedly continuous on the closed half-plane $\{s \in C: \ \mathrm{Re}\, s \ge 0\}$. Moreover,

$$\lim_{\substack{|s|\to\infty \\ -\frac{\pi}{2}\le \arg s \le \frac{\pi}{2}}} \left\{ e^{st} \frac{\rho^{[t]+1}(s)}{1-\rho(s)} \right\} = 0 .$$

Hence, by Cauchy's theorem,

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\rho^{[t]+1}(s)}{1-\rho(s)} \frac{ds}{s+\lambda} = 0 .$$

On the other hand,

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \frac{1}{2} \sum_{k=1}^{[t]} \rho^k(s) \frac{ds}{s+\lambda} = \sum_{k=1}^{[t]} 2^{k-1} \lambda^k e^{-\lambda k} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{s(t-k)} \frac{ds}{(s+\lambda)(s+2\lambda)^k} . \tag{4}$$

An application of the residue theorem reveals that for $k = 1,\ldots,[t]$ and $t \ge 1$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s(t-k)}\frac{ds}{(s+\lambda)(s+2\lambda)^k} = \left.\frac{e^{s(t-k)}}{(s+2\lambda)^k}\right|_{s=-\lambda} + \frac{1}{(k-1)!}\left(\frac{\partial}{\partial s}\right)^{k-1}\left.\frac{e^{s(t-k)}}{s+\lambda}\right|_{s=-2\lambda}.$$

However, by Leibniz's formula,

$$\left(\frac{\partial}{\partial s}\right)^{k-1}\frac{e^{s(t-k)}}{s+\lambda} = \sum_{j=0}^{k-1}\binom{k-1}{j}\left(\frac{\partial}{\partial s}\right)^{k-1-j}(s+\lambda)^{-1}\left(\frac{\partial}{\partial s}\right)^{j} e^{s(t-k)}$$

$$= \sum_{j=0}^{k-1}\binom{k-1}{j}(k-1-j)!(-1)^{k-1-j}(s+\lambda)^{j-k}(t-k)^j e^{s(t-k)}.$$

Hence,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{s(t-k)}\frac{ds}{(s+\lambda)(s+s\lambda)^k} = \lambda^{-k}e^{-\lambda(t-k)}\left(1-e^{-\lambda(t-k)}\sum_{j=0}^{k-1}\frac{(\lambda(t-k))^j}{j!}\right). \qquad (5)$$

By (3), (4), (5) and the identity

$$\sum_{k=1}^{[t]} 2^{k-1} = 2^{[t]} - 1,\ t \geq 0,$$

we finally obtain

$$\Re(t) = e^{-\lambda t}\left\{2^{[t]} - \sum_{k=1}^{[t]} 2^{k-1} e^{-\lambda(t-k)}\sum_{j=0}^{k-1}\frac{(\lambda(t-k))^j}{j!}\right\},\ t \geq 0.$$

Figure 1 shows the graph of $\Re(.)$, $\lambda = 0.5$ (case 1) and $\lambda = 0.3$ (case 2).

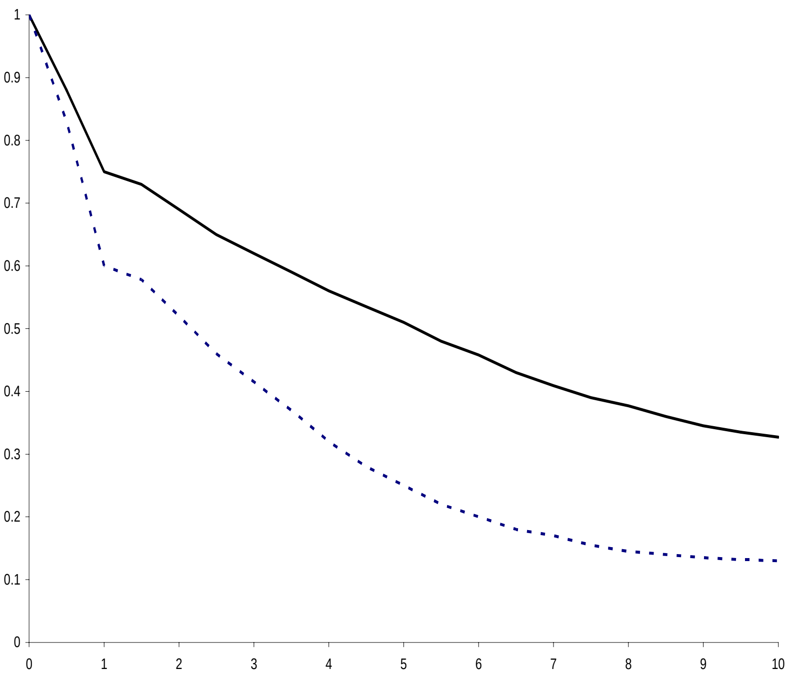

Figure 1: Graph of $\Re(.)$, $\lambda = 0.3$ (upper graph), $\lambda = 0.5$ (lower graph).

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