INFORMATION RECYCLING MATHEMATICAL METHODS FOR PROTEAN SYSTEMS: A PATH-WAY APPROACH

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"The world can change in a day, all right, but not always the way we think it will"[2]

ABSTRACT

In this paper, a concept of information recycling mathematical methods has been introduced for mathematical modelling and analysis of problems arising in real life systems. Based on the proposed concept, an algorithm for solving a linear programming model arising in a protean environment has been discussed. This concept has been extended to consider the case of a quadratic programming model and a differential equation of the second order. It has been pointed out that the proposed concept is a challenge to reconsider other mathematical modelling and solution procedures in a protean environment.

OPSOMMING

Die beginsel van sikliese behandeling van inligting met behulp van wiskundige metodes en modelle word voorgehou vir die ontleding van praktykprobleme. Gebaseer op hierdie gedagte word 'n algoritme bespreek vir die oplossing van 'n liniëre programmeringsmodel in 'n Preusagtige omgewing. Hierna word die metode uitgebrei vir 'n kwadratiese programmeringsmodel en 'n tweede-orde differensiaalvergelyking. Die resultate van die ondersoek dien as 'n rigtingswyser vir ondersoeke van oplossingprosedures waar 'n veranderende omgewing ter sprake is.

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1. INTRODUCTION

Consider an operating system under a protean environment where input in a model keep changing due to external and internal interactions [11, 12, 13]. This kind of situation arises in industry or any other real-life application where mathematical models have been developed for analysis and understanding of a physical process. In this changing environment, for obtaining new results to the changed problem, it may be desirable to make use of information and analytical results that are already available from the solution and analysis that was carried out before changes were experienced. This kind of thinking is labelled by the author as a 'path-way' approach and all such methods that take advantage of the existing information as "information recycling mathematical methods". In order to keep the presentation simple and meaningful, a linear programming model has been used to explain the proposed concept and solution procedure. A few more simple cases have been cited where the suggested approach seems feasible and may have some computational advantage. The suggested approach, in the opinion of this author, is a philosophy and a mathematical challenge to investigate associated advantages, disadvantages and limitations. Since ideas presented in this paper are general, an approach on similar lines may be desirable in other situations. This kind of approach, may have scope for further research and investigations.

The paper has been organised in the following sections. In section 2, terms in the context of the present paper have been explained. Section 3 deals with the philosophy and the associated challenge. In section 4, a mathematical statement of the problem considered in this paper has been discussed. In section 5, a LP model has been used as a vehicle for developing an 'Information Recycling Path-way Approach' for the LP model used heavily by the industry. These ideas have been extended to a protean quadratic programming model (QPP) in section 6. An ordinary second order differential equation has been considered in section 7 and finally in section 8, this discussion has been concluded by some remarks and direction for further research work.

2. CLARIFICATION OF TERMS

Terms such as 'recycling', 'protean system' and 'path-way approach', used above are explained in the context of the present paper.

2.1 Information recycling

Recycling is an accepted concept in waste management. It is a known fact that society has learnt and accepted recycling as an integral part of daily life after wasting lots of natural resources. Recycling has been motivated on two accounts, the first objective is to reduce the bulk of solid waste, and the second objective is to convert solid waste into a useful resource. Thus this 'waste-to-resource' is a journey that has taken long in reference to waste management and has hardly been used in the context of mathematics or mathematical computations. The concept of information recycling can be identified in many mathematical methods, for example, in solving recurrence equations in dynamic programming [1, 2, 3]. Solid waste or industrial waste is now considered as a transient state until some usefulness is discovered. For an interesting discussion see Grover *et al* [7]. Disposal of solid and industrial waste in the environment is no longer an acceptable solution. This fact has been learnt by society by paying the price in terms of spoiling many natural resources. Similarly, at present, computer speed and capability of handling calculations is so high that it may seem to be an infinite supply but it would be desirable to learn to conserve computational resources as much as possible. It is in this context that some existing methods are re-looked at from this new point of view, which is called 'information recycling methods'.

2.2 Protean system

The concept of information recycling is more appropriate in a 'protean environment' [11, 12] where input to a model might change but broadly speaking model structure may remain valid in a given circumstance. The term protean is used to reflect an environment that changes. Changes in any real-life system are natural. These changes might be experienced due to interaction with other internal and external organizations [11]. If a mathematical model is useful in this kind of protean environment for management decisions, it is reasonable to think that input data might change from one period to another but the model may remain applicable. For example, a population model may be described by a differential equation, but its input parameters are subject to change due to interaction from internal components and external systems. Medical research and government policy can easily alter values of infant survival rate. This 'repeated use of a similar model with changes in its input data has been described as a protean environment'. The objective of this paper is to promote 'information recycling methods' for solving mathematical models arising in a protean environment.

2.3 Path-way solution approach

Finally, the term 'path-way' approach has been used. In a path-way solution approach, a path is created that joins a solution of one problem to a solution of another problem which in model structure is essentially the same but is different with respect to its input parameters, see Garcia and Zangwill [6]. This is a new direction for solving problems that arise in a protean environment. It has been assumed that a particular mathematical model remains valid but its input parameters change due to its interaction with other systems. This is called a 'path-way method.'

In section 4, a formal statement of the problem considered in this paper is presented. For this purpose a linear programming model has been used and necessary ideas explained. It is the expectation of the author that recycling information may result in more efficient methods and reduced computational efforts required to solve the problems arising in a protean environment. Present work reported in this paper is not sufficient to make a meaningful conclusion, however, further work and insight in computational aspect is desirable and may prove exciting for real life applications.

3. MATHEMATICAL CHALLENGE AND PHILOSOPHY

In this section justification is provided for the challenge created by the above approach and for the philosophy behind it.

3.1 The challenge

Each mathematical method is based on some assumptions that must be satisfied before its application is justified. For example, in the case of an LP model, an application of the simplex method assumes existence of an extreme point starting solution before the simplex search for an optimal solution begins. This extreme point condition is either easily satisfied by the given set of constraints or it is artificially forced by creating a required number of artificial variables. However, in consideration of a protean case, although the optimal solution to the problem in the i^{th} period would satisfy the requirement of the extreme point solution but would not necessarily hold the extreme point condition of the current optimal solution with regard to the changed input in the $(i+1)th$ interval. Thus one has to create a path starting from a point that may not be an extreme point to first approach an extreme point of the new convex set with respect to the changed input. This requirement demands new ideas and possibly new procedures to overcome the situation. Thus mathematical challenges would be created when the proposed approach is applied.

3.2 The philosophy

The dictionary meaning of the word 'philosophy' is 'any investigation of natural phenomena' or 'love for wisdom'. The proposed approach suggests an alternative approach to problem solving, particularly for the protean case. An easy option would always be to reapply the standard method to solve the changed problem with respect to the changed data, and obtain a new answer. But the investigation with regard to using the existing information to reach a solution to the changed problem is a search for wisdom and advantages. Thus the proposed approach justifies usage of the word philosophy.

Thus, in the opinion of the author, the proposed approach is a mathematical challenge and a philosophy for developing new mathematical ideas.

4. MATHEMATICAL STATEMENT OF INFORMATION RECYCLING

Conventional discussion on sensitivity and post-optimality analysis of a mathematical model is somewhat similar to information recycling, however, in those discussions generally it is assumed that changes take place in a specific way. Such an assumption is unrealistic as the root cause of a change is an interaction of a given system with other external systems and components within the system. For example, in an LP model changes can arise simultaneously in the values of c_j, b_i, a_{ij} and even an extra constraint or a variable may be justifiably added or deleted. Changes may be periodic, like each day, week, or each month etc. It is unlikely that input data will remain unchanged in a situation where the same model may be applicable again and again. In this paper, the concept of recycling from the waste management is used to recycle a part of the mathematical information available from the optimal solution to the i^{th} interval model. Mathematically the problem is to investigate what part of the information is invariant with regard to the changed problem in the $(i+1)th$ interval. Thus in summary, **'**a mathematical problem of interest is to find a path joining a solution of a given model before changes to its new solution after changes**'**. Note that it has been assumed that input information to a model is changing frequently but model structure remains valid.

5. INFORMATION RECYCLING IN AN LP MODEL

Consider an industrial organization using an LP model to find the optimal mix on some regular intervals, which may be any predetermined time period represented by a day or a week etc. Let the LP model be:

Maximize $x_0 = CX$,
subject to $AX \leq b, X \geq 0.$ (1)

Here A, b, C are assumed known values and vector $X \geq 0$ is the unknown to be determined. Since changes may arise, values of A, b, C in two consecutive intervals are unlikely to remain the same. Hence the LP model in the i^{th} interval and the $(i+1)th$ interval are as follows:

Maximize $x_0^i = C_i X_i$
Subject to
$A_i X_i \leq b_i$
$X_i \geq 0$ (2)

and

Maximize $x_0^{(i+1)} = C_{i+1} X_{i+1}$
Subject to
$A_{i+1} X_{i+1} \leq b_{i+1}$
$X_{i+1} \geq 0$ (3)

Here it is assumed that an optimal solution to the problem (2) i.e. solution of the model for the i^{th} interval has been obtained by using an appropriate LP solution procedure [4, 10]. The problem (3) for the next interval, requires attention if some or all values of various elements in A, b, C have changed. Thus the problem considered here is to find an optimal solution to problem (3) from the optimal solution already obtained to the problem (2). In other words, find the path that joins the optimal solution of the problem (2) with an optimal solution of the problem (3). This paper is addressing the path-way solution philosophy with regard to an LP model, but an appropriate approach for other type of mathematical analytical tools will be the subject matter of subsequent publications. Based on the protean philosophy under a changing environment an algorithm was developed and reported in an earlier paper by Talukder and Kumar [15].

5.1 A three-stage path way approach to the LP model

The aim is to develop an approach to find a path joining the optimal solution of problem (2) to the optimal solution to problem (3). It is assumed that the new mathematical problem for the $(i+1)th$ interval is also an LP model; where changes may arise with respect to the values of some or all elements of A,b,C. This is a realistic assumption, as no management can fully control changes that may arise in various inputs of a system as an on going process. However, it is certain that the changed values can be expressed by simple addition and subtraction with respect to previous values. Thus one can write the new values of the elements A,b,C in the form of the old values of those elements as follows:

$$
\begin{aligned}
c_j^{i+1} &= c_j^i + \overline{c_j^i}, \forall j, \\
a_{ij}^{i+1} &= a_{ij}^i + \overline{a_{ij}^i}, \forall ij, and \\
b_i^{i+1} &= b_i^i + \overline{b_i^i}, \forall i
\end{aligned} \tag{4}
$$

Note that $i = 1,2,...m$; $j = 1,2,...,n.$ and $\overline{c_j^i}, \forall j, \overline{a_{ij}^i}, \forall ij, \overline{b_i^i}, \forall i$ are all unrestricted positive, negative or zero known values. Thus the problem (3) can be expressed in two parts. In this presentation, the part one is identical to the problem (2) and the remaining residual values form the part two of the modified model, which is likely to be a sparse problem. Thus using the relations (4), the problem (3) can be rewritten as:

$$
\begin{aligned}
&\text{Maximize } x_0^{(i+1)} = C_i X_i + \overline{C_i}\,\overline{X_i} \\
&\text{Subject to} \\
&A_i X_i + \overline{A_i}\,\overline{X_i} \le b_i + \overline{b_i} \\
&X_i \ge 0, \overline{X_i} \ge 0.
\end{aligned} \tag{5}
$$

Thus with respect to the i^{th} interval model (2), an additional sparse structure has been added to it. In other words, it is anticipated that the modified sparse part II of the model would be given by:

$$
\begin{aligned}
&\text{Maximize: } \overline{C_i}\,\overline{X_i} \\
&\text{Subject to} \\
&\quad \overline{A_i}\,\overline{X_i} \le \overline{b_i} \\
&\quad \overline{X_i} \ge 0.
\end{aligned} \tag{6}
$$

Here for simplicity, it has been assumed that $n_{i+1} = n_i$ and $m_{i+1} = m_i$, then the same basis as for the i^{th} interval model can be used on the non-zero columns of sparse model (6) to obtain the corresponding equivalent simplex matrix with respect to the basic elements in the i^{th} interval problem. Once values for the sparse problem (6) have been obtained, it is realised that elements, $c_j^i, \overline{c_j^i}$; $a_{ij}^i, \overline{a_{ij}^i}; b_i^i, \overline{b_i^i}$ are not independent. For example, with respect to the $(i+1)^{th}$ interval model, columns have meaning and interpretation only when each pair of elements $c_j^i, \overline{c_j^i}$; $a_{ij}^i, \overline{a_{ij}^i}$ and $b_i^i, \overline{b_i^i}$ are reduced back to a single element as shown by the relation (4). Thus the combined problem may not be able to hold its simplex structure with respect to basis, extreme point representation, feasibility and optimality for the problem under investigation i.e. the problem for the $(i+1)^{th}$ interval. In other words, it has been emphasized that the resulting solution with respect to the $(i+1)^{th}$ interval problem may not even be an extreme point solution. At this stage a three- stage algorithm to restore characteristics of the simplex approach and to find the new optimal solution can be proposed as follows:

Stage 1:

Perform pivoting operations to restore the basis, i.e. for each basis variable in the i^{th} interval, a unit column prevails for the data with respect to the $(i+1)^{th}$ interval.

Stage 2:

After stage 1 operations, if the optimality condition has been lost, use the normal simplex iterations to establish the optimality conditions for the $(i+1)^{th}$ interval.

Stage 3:

In the resulting tableau after stage 2 computations, if the solution does not satisfy feasibility requirement, use the dual simplex method to obtain a feasible solution, while holding to the optimality conditions.

At end of the stage 3 computations, the final result obtained in the required optimal solution to the problem for the $(i+1)^{th}$ interval.

Note it was assumed that $n_{i+1} = n_i$ and $m_{i+1} = m_i$, which is not an essential requirement but was assumed for simplification and presentation of the three-stage algorithm. In case this condition is not satisfied, one has to carry out some modifications as explained in the algorithmic steps of the proposed method.

Thus starting from the optimal solution to the i^{th} interval problem one can reach the optimal solution to the $(i+1)^{th}$ interval problem. It may also be noted that each tableau in the three stage calculations represents a point in an n dimensional space, which forms a piece-wise linear path joining the optimal solution of the problem at the i^{th} interval to the optimal solution of the problem at the $(i+1)^{th}$ interval.

5.2 The algorithm

The steps of the algorithm are as follows:

Step 0

Record for the i^{th} interval LP model values of

$$n_i, m_i$$
$$c_j^i \forall j, a_{ij}^i \forall ij, b_i^i \forall i$$
$$x_{Bl}^i, l = 1,2,...,m$$

and the final simplex tableau giving an optimal solution.

Record the problem data for the $(i+1)^{th}$ interval problem i.e. values of:

$$n_{i+1}, m_{i+1}$$
$$c_j^{i+1} \forall j, a_{ij}^{i+1} \forall ij, b_i^{i+1} \forall i$$
$$x_{Bl}^i, l = 1,2,...,m$$

Step 1

Compare whether all variables in the i^{th} interval model are identical to the model for the $(i+1)^{th}$ interval. If so pass on to step 2. Otherwise carry out the following modifications:

If a variable present in the model for the i^{th} interval is no more required in the model for the $(i+1)^{th}$ interval, create a duplicate column with negative values for the sparse problem 6 and substitute as given in (7).

$$\begin{aligned} \overline{c_j^i} &= -c_j^i \\ \overline{a_{ij}^i} &= -a_{ij}^i \end{aligned} \qquad (7)$$

If a new variable x_{n+1}^{i+1} is required in the $(i+1)^{th}$ interval problem compared to the i^{th} interval model, it is simply added to the sparse problem 6 and substitute:

$$\begin{aligned} \overline{c_{n+1}^i} &= c_{n+1}^{i+1} \\ \overline{a_{i,n+1}^i} &= a_{i,n+1}^i \end{aligned} \qquad (8)$$

Step 2

Compare constraints in the i^{th} interval problem and the model for the $(i+1)^{th}$ interval. If the number of constraints is equal in two consequent interval models, we move on to the step 3. Otherwise carry out the following modification, as may be appropriate:

Consider that the l^{th} constraint is no longer required in the model representing the problem in the $(i+1)^{th}$ interval, simply substitute:

$$\overline{b_l} = M \ , \tag{9}$$

where M has a usual meaning representing a large numerical value to make this constraint redundant.

A new $(m+1)^{th}$ constraint has to be added to the model of the problem for the $(i+1)^{th}$ interval, which did not exist when the model for the i^{th} interval model was developed. Thus to add this new constraint one is required to carry out a modification and substitution as follows:

Modification: Add as a $(m+1)^{th}$ row and column to an existing mXm matrix a row and a column of a unit matrix of the i^{th} interval problem, and substitute

$$\begin{aligned} b^i_{m+1} &= 0 \\ \overline{b^i_{m+1}} &= b^{i+1}_{m+1} \\ \overline{a^{\,i}_{j,m+1}} &= a^{\,i+1}_{j,m+1} \end{aligned} \tag{10}$$

Step 3

Rearrange the problem for the $(i+1)^{th}$ interval into two parts, as shown in relation (5). If the sparse part of the problem does not exist, the optimal solution for the i^{th} interval remains optimal for the $(i+1)^{th}$ interval. Go to step 6; otherwise go to step 4.

Step 4

With respect to the basis of the model for the i^{th} interval, perform $B_i^{-1}.\overline{P_j}$ calculations on each column $\overline{P_j}$ of the sparse part of the problem and combine the corresponding column j with $\overline{j}$. Carry out pivot operations so that the basis with respect to the i^{th} interval is restored. If the resulting solution does not satisfy feasibility condition, use dual simplex iterations to restore the feasibility of the new solution. When feasible solution has been obtained, go to step 5.

Step 5

Since a basic feasible solution has been obtained, carry out simplex iterations to obtain an optimal solution. Go to step 6.

Step 6

The current solution is optimal for the $(i+1)^{th}$ interval. Print the optimal solution. For the next interval, assign $i+1=i$ and go to Step 0.

5.3 LP illustration

Consider the following example as an LP model for the i^{th} interval.

Example 1.

Maximize $x_0 = 3x_1 + 2x_2 + 2x_3$
Subject to

$$\begin{aligned} &x_1 + 3x_2 + x_3 \le 16 \\ &2x_1 + x_2 + 2x_3 \le 18 \\ &x_j \ge 0,\ j = 1,2,3. \end{aligned} \tag{11}$$

Let the optimal solution be given by

ROW (BASIS)	X1	X2	X3	SLK 2	SLK3	RHS
1 Obj Row	0.000	0.000	1.000	0.200	1.400	28.400
2	0.000	1.000	0.000	0.400	-0.200	2.800
3 X1	1.000	0.000	1.000	-0.200	0.600	7.600

Consider that the problem in the $(i+1)^{th}$ interval become

Maximize $x_0 = 3x_1 + 1x_2 + 3x_3$
Subject to

$$\begin{aligned} &2x_1 + 3x_2 + 2x_3 \le 16 \\ &2x_1 + x_2 + 2x_3 \le 14 \\ &x_j \ge 0,\ j = 1,2,3. \end{aligned} \tag{12}$$

Rewriting this in two parts by using relation (4), we get

Maximize $x_0 = 3x_1 + 2x_2 + 2x_3 + 0\overline{x_1} - \overline{x_2} + \overline{x_3}$
Subject to

$$\begin{aligned} & x_1 + 3x_2 + x_3 + \overline{x_1} + 0\overline{x_2} + \overline{x_3} \le 16 + 0 \\ & 2x_1 + x_2 + 2x_3 + 0\overline{x_1} + 0\overline{x_2} + 0\overline{x_3} \le 18 - 4 \\ & x_j \ge 0,\ j = 1,2,3. \end{aligned} \tag{13}$$

With respect to basis $X_B = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$, we can write (13) from the solution of (11) as:

ROW	(BASIS)	X1	X2	X3	$\overline{x_1}$	$\overline{x_2}$	$\overline{x_3}$	SLK 2	SLK 3	RHS
1	Obj Row	0.000	0.000	1.000				0.200	1.400	28.400
2	X2	0.000	1.000	0.000				0.400	-0.200	2.800
3	X1	1.000	0.000	1.000				-0.200	0.600	7.600

The three missing columns can be computed from the solution matrix (11) and the corresponding inverse matrix with respect to the basic variables. These three columns with respect to variables $\overline{x_1}$, $\overline{x_2}, \overline{x_3}$; are: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ respectively. The corresponding values of the missing three columns are given by $B^{-1}P_i$, where $i = 1,2,3$ are the above three columns and the value of B^{-1} is obtained from the solution of the problem for the i^{th} interval (11). Thus the missing values respectively are: $\begin{bmatrix} 0.2 \\ 0.4 \\ -0.2 \end{bmatrix}, \begin{bmatrix} -1.0 \\ 0.0 \\ 0.0 \end{bmatrix}, \begin{bmatrix} 1.2 \\ 0.4 \\ -0.2 \end{bmatrix}$

Thus the new tableau in respect of the modified problem would be given by:

ROW	(BASIS)	X1	X2	X3	$\overline{x_1}$	$\overline{x_2}$	$\overline{x_3}$	SLK 2	SLK 3	RHS
1	Obj Row	0.00	0.00	1.00	0.2	-1.0	1.2	0.200	1.400	28.400-5.6
2	X2	0.000	1.00	0.00	0.4	0.0	0.4	0.400	-0.200	2.800+0.8
3	X1	1.00	0.00	1.00	-0.2	0.0	-0.2	-0.200	0.600	7.600-2.4

Using the relation (4), we combine columns of variables $x_1, \overline{x_1}; x_2, \overline{x_2}; x_3, \overline{x_3}$ and obtain,

ROW	(BASIS)	X1	X2	X3	SLK 2	SLK 3	RHS
1	Obj Row	0.200	1.000	2.200	0.200	1.400	22.8
2	X2	0.400	1.000	0.400	0.400	-0.200	3.60
3	X1	0.800	0.000	0.800	-0.200	0.600	5.20

Note that columns with respect to the basic variables $x_1; x_2$, are no longer unit columns. Consequently, first restore these columns as a unit column and then obtain the optimal solution. The resulting two tableaus are:

ROW	(BASIS)	X1	X2	X3	SLK 2	SLK 3	RHS
1	Obj Row	0.0	0.0	2.0	-0.25	1.75	20.5
2	X2	0.0	1.0	0.0	0.5	-0.5	1.0
3	X1	1.0	0.0	1.0	-0.25	0.75	6.5

and

ROW	(BASIS)	X1	X2	X3	SLK 2	SLK 3	RHS
1	Obj Row	0.0	0.5	2.0	0.0	1.5	21.0
2	S2	0.0	2.0	0.0	1.0	-1.0	2.0
3	X1	1.0	0.5	1.0	0.0	0.5	7.0

The resulting new solution is given by: $x_0 = 21, x_1 = 7, SLK2 = 2.0$ as basic variables and other variables are zero as non-basic variables.
Thus the path joining the optimal solution $(x_1, x_2, x_3, s_1, s_2) = (7.6,2.8,0.0,0.0,0.0) \Rightarrow (6.5,1.0,0.0,0.0,0.0) \Rightarrow (7.0,0.0,0.0,0.0,2.0)$.
The values of the objective function have moved from $x_0 = 28.4$ to $x_0 = 21$.

5.4 Discussion and remarks

The suggested approach uses ideas of parametric programming, but it is not a parametric approach. It uses a dichotomy to relate present values in relation to the immediate past.

By using the above approach discussed in this paper, it may be concluded that optimal solutions to all LP's are connected by a piece-wise linear path joining them, and this path may not pass through extreme points in the feasible region. However, if changes are recorded in each column, the computational load will increase; consequently it may not be wise to find the new solution by the path approach. If changes are relatively small, this approach may prove to be worthwhile. For further testing, necessary software support is desirable to establish its computational efficiency.

6. A QUADRATIC PROGRAMMING (QP) MODEL IN A PROTEAN ENVIRONMENT

6.1 Standard QP and conditions for optimality

Consider a quadratic programming problem (QPP):

Maximize $x_0 = CX + X^t DX$

Subject to $G(X) \equiv \begin{bmatrix} A \\ -I \end{bmatrix} X - \begin{bmatrix} b \\ 0 \end{bmatrix} + \begin{bmatrix} U^2 \\ V^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (14)

Let Lagrange multipliers corresponding to the constraint conditions $AX \leq b$ and $X \geq 0$ be denoted by $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)^t$ and $\mu = (\mu_1, \mu_2, ..., \mu_n)^t$ respectively. The Kuhn-Tucker (KT) optimality conditions for the optimal solution are given by four sets of conditions as given below:

Set 1:

$$\frac{\partial L}{\partial x_j} = \nabla x_0 + (\lambda^t, \mu^t)\nabla G(X) = 0,\ j = 1,2,...n. \quad (15)$$

Set 2:

$$\left.\begin{matrix} \lambda_i u_i = 0 \\ x_j v_j = 0 \end{matrix}\right] \Rightarrow \left.\begin{matrix} \lambda_i (\sum_{j=1}^{n} a_{ij} x_j - b_i) = 0 \\ \mu_j x_j = 0 \end{matrix}\right] \quad (16)$$

Set 3:

$$AX \leq b,\ X \geq 0 \quad (17.1)$$

Set 4:

$$\lambda \geq 0, \mu \geq 0, u_i^2 = s_i \geq 0 \quad (17.2)$$

A feasible solution to the above conditions (15), (17.1) and (17.2) can easily be obtained by an application of the simplex like method except for condition (16), which is a set of linear complementarity conditions. Wolfe's procedure uses a restricted base entry in the simplex method to satisfy these complementary conditions. Under the protean environment similar to one we used for the linear programming problem in section 5, the protean QPP can easily be approached by the path-way method.

6.2 Protean QP problem

Consider the QP models in the i^{th} interval and the $(i+1)th$ interval are represented as follows:

Maximize $x_0^i = C_i X_i + X_i^T D_i X_i$
Subject to

$$A_i X_i \le b_i \qquad X_i \ge 0 \tag{18}$$

And

Maximize $x_0^{(i+1)} = C_{i+1} X_{i+1} + X_{i+1}^T D_{i+1} X_{i+1}$
Subject to

$$A_{i+1} X_{i+1} \le b_{i+1} \qquad X_{i+1} \ge 0 \tag{19}$$

Once again, the problem (19) may be expressed partly in the form of (18) and the rest as a sparse problem by using the relations:

$$\begin{aligned} c_j^{i+1} &= c_j^i + \overline{c_j^i}, \forall j, \\ D_{i+1} &= D_i + \overline{D_i} \\ a_{ij}^{i+1} &= a_{ij}^i + \overline{a_{ij}^i}, \forall ij, and \\ b_i^{i+1} &= b_i^i + \overline{b_i^i}, \forall i \end{aligned} \tag{20}$$

where elements in $\overline{C_i}, \overline{D_i}, \overline{A_i}, \overline{b_i}$ can be positive or negative as they are governed by the changes that may arise from i^{th} interval to $(i+1)th$. Thus optimality conditions for the QP (19) are:

$$\begin{aligned} \tfrac{\partial L}{\partial x_j} &= \nabla x_0^{i+1} + (S_{i+1}^{+t}, SX_{i+1}^{+t}) \nabla G_{i+1}(X) = 0,\ j = 1,2,...n. \\ A_{i+1} X_{i+1} &\le b_{i+1}, \end{aligned} \tag{21}$$

In addition, all variables are restricted to non-negative values.

Thus the approach discussed for a protean LP is again applicable for a protean QLP. This is illustrated by a numerical example given below.

6.3 Numerical illustration

Example 2

Consider that a QP problem for the i^{th} interval is given by:
Minimize $x_0 = 2x_1^2 + 2x_1 x_2 + 2x_2^2 - 4x_1 - 6x_2$
Subject to

$$x_1 + 3x_2 \leq 3$$
$$x_1, x_2 \geq 0 \tag{22}$$

Let the optimal solution to (22) by using the KT conditions be given in Table 1:

	x_1	x_2	λ	μ_1	μ_2	R_1	R_2	S	RHS
$x_0^{'}$	0	0.0	0.0	0.0	0.0	1.0	0.0	0.0	0
x_1	1.0	0.0	0.0	-9/28	3/28	9/28	-3/28	-1/14	3/7
λ	0.0	0.0	1.0	1/14	-5/14	-1/14	-5/14	-6/14	4/7
x_2	0.0	1.0	0.0	3/28	-1/28	-3/28	1/28	5/14	6/7

Table 1: Optimal solution to QP (22)

Consider that for the $(i+1)^{th}$ interval the problem got modified as given by (23).

Minimize $x_0 = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1$
Subject to

$$x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0 \tag{23}$$

Using relation (20), the parametric problem expressed in the form of problem (22) would be given by:

Minimize $x_0 = 2x_1^2 + 2x_1x_2 + 2x_2^2 - 4x_1 - 6x_2 + t(-4x_1x_2 - 2x_1 + 6x_2)$
Subject to

$$x_1 + 3x_2 - 2tx_2 \leq 3 - t$$
$$x_1, x_2 \geq 0 \tag{24}$$

Note that (24) at t=0 becomes (22) and at t=1, it becomes (23). The Lagrange function for problem (24) is given by:

$$L = [2x_1^2 + 2x_1x_2 + 2x_2^2 - 4x_1 - 6x_2 + t(-4x_1x_2 - 2x_1 + 6x_2)] +$$
$$\lambda[x_1 + 3x_2 + s - 3 + t(2x_2 + 1)] + \mu_1(-x_1 + v_1^2) + \mu_2(-x_2 + v_2^2) \tag{25}$$

The KT conditions from (25) would result in three more variables tx_1, tx_2 and $t\lambda$. One can easily verify that the Table 1 will remain valid with respect to all variables except the three new variables just mentioned above. For these three variables, one can obtain the corresponding columns by the usual inverse matrix and column multiplication. The final result is given in Table 2.

	x_1	tx_1	x_2	tx_2	λ	$t\lambda$	μ_1	μ_2	R_1	R_2	S	RHS
$x_0^{'}$	0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	1.0	0.0	0+t(RHS)
x_1	1.0	3/7	0.0	-8/7	0.0	3/14	-9/28	3/28	9/28	-3/28	-1/14	3/7+19/14
λ λ	0.0	-10/7	0.0	8/7	1.0	-5/7	1/14	-5/14	-1/14	5/14	-3/7	4/7-13/7
x_2	0.0	-1/7	1.0	-2/7	0.0	-1/14	3/28	-1/28	-3/28	1/28	5/14	6/7- 11/14

Table 2: Values directly obtained from matrix 1 and matrix inverse multiplication.

For t=1, the problem reduces to the one we wish to solve for the $(i+1)^{th}$ interval. Hence put t=1 and combine the columns 1 and 2 as the column for the variable x_1, similarly columns 3 and 4 are combined to form a column for the variable x_2 and columns 5 and 6 form the column for the variable λ. This new information is given in Table 3.

	x_1	x_2	λ	μ_1	μ_2	R_1	R_2	S	RHS
$x_0^{'}$	0	0.0	0.0	0.0	0.0	1.0	1.0	0.0	
x_1	10/7	-8/7	3/14	-9/28	3/28	9/28	-3/28	-1/14	25/14
λ λ	-10/7	8/7	2/7	1/14	-5/14	-1/14	5/14	-3/7	-9/7
x_2	-1/7	5/7	-1/14	3/28	-1/28	-3/28	1/28	5/14	1/14

Table 3: Non-basic matrix with respect to the new problem

Since the unit column nature with respect to basic variables has been lost, it is restored again by pivoting with respect to x_1, λ and x_2 in row 1, row 2 and row 3 respectively. This will result in the Table 4.

	x_1	x_2	λ	μ_1	μ_2	R_1	R_2	S	RHS
$x_0^{'}$	0	0.0	0.0	0.0	0.0	1.0	1.0	0.0	
x_1	1	0.0	0.0	-1/12	1/12	1/12	-1/12	1/2	3/2
λ λ	0.0	0.0	1.0	-1/2	-1/2	1/2	1/2	-1	1
x_2	0.0	1.0	0.0	1/12	-1/12	-1/12	1/12	1/2	½

Table 4: Solution to the new problem.

This solution is optimal, as optimality conditions are satisfied, and it is feasible. The path it has followed is:

$(x_1 = 3/7, x_2 = 6/7) \rightarrow (x_1 = 25/14, x_2 = 1/14) \rightarrow (x_1 = 5/4, x_2 = 5/12) \rightarrow (x_1 = 19/12, x_2 = 5/12) \rightarrow$
$(x_1 = 3/2, x_2 = 1/2)$.

This process can be further simplified by using a reformulation discussed in section 4.4.

6.4 Reformulation of QPP – a two-in-one characteristic variable

Define a new slack variable $U^2 = (s_1, s_2, ..., s_m)^t$, such that

$$s_i^+ = \max[0, s_i]$$
$$s_i^- = \max[0, -s_i] \quad (26)$$

where let $s_i = \sum_{j=1}^{n}(a_{ij}x_j - b_i)$ and thus in the infeasible region $s_i \geq 0, s_i^+ \geq 0$ and $s_i^- = 0$. Similarly in the feasible region $s_i \leq 0, s_i^- \geq 0$ and $s_i^+ = 0$. Also it may be observed that for all values of s_i, the product $s_i^+ . s_i^- = 0$. Therefore, the Lagrange Multiplier λ_i in relation (15) can be replaced by s_i^+ and condition (16) is satisfied by the nature of the function. Similarly the Lagrange multiplier μ_j for the inequalities $X \geq 0$ can be replaced by another slack variable sx_j^+, where $sx_j^- = \max[0, -x_j]$. Thus for the inequalities $X \geq 0$, the other set of complementarity $\mu_j x_j = 0$ are satisfied when μ_j's are replaced by sx_j^+. Thus KT conditions (15, 16, 17) for optimality can be replaced by:

$$\tfrac{\partial L}{\partial x_j} = \nabla x_0 + (S^{+t}, SX^{+t})\nabla G(X) = 0,\ j = 1,2,...n.$$
$AX \leq b$, and all variables are non-negative. (27)

Note that $S^{+t} = (s_1^+, s_2^+, ..., s_m^+)^t$ and $SX^{+t} = (sx_1^+, sx_2^+, ..., sx_n^+)^t$.

Reconsider the QPP (22) and (23) and the corresponding Lagrange function as defined earlier would be given by:

$$L = [2x_1^2 + 2x_1x_2 + 2x_2^2 - 4x_1 - 6x_2 + t(-4x_1x_2 - 2x_1 + 6x_2)] +$$
$$\max[0, s][s - (x_1 + 3x_2 - 3 + t(2x_2 + 1))] + \max[0, sx_1](-x_1 + sx_1) + \max[0, sx_2](-x_2 + sx_2)$$

This will result in the following KT conditions:

$$\frac{\partial l}{\partial x_1} = 4x_1 + 2x_2 - 4 - 4tx_2 - 2t - \max[0, s] - \max[0, sx_1] = 0$$

Similar condition can be written for the variable x_2. Other conditions would be the given set of constraints obtained by differentiating the Lagrange function with respect to the Lagrange multipliers.

7. SOLUTION OF A SECOND ORDER DIFFERENTIAL EQUATION (DE)

For the sake of applying the proposed approach to other situations, consider a second order DE. The characteristic function of such a DE is a quadratic equation of the form $ax^2 + bx + c = 0$, which are solved to obtain the general solution. The two roots of this equation x_1, x_2 are given by $\{-b \pm \sqrt{(b^2 - 4ac)}\}/2a$. However, the situation is such that values of the constants a,b,c keep changing due to external interactions with the system. Although for each set of values of a,b,c the roots determination formula given above can be used to find new roots, however, the aim here is to illustrate the path-way approach for these system of quadratic equations. Here consider that at time i, the characteristic equation of the given DE is given by the quadratic equation $a_i x^2 + b_i x + c_i = 0$ and at time $i+1$, the characteristic equation becomes $a_{i+1}x^2 + b_{i+1}x + c_{i+1} = 0$. Once again, assume that the relationship between a,b,c at time i and $i+1$ is given by relations similar to equation (4). In other words:

$$\begin{aligned} a_{i+1} &= a_i + \Delta a_i, \\ b_{i+1} &= b_i + \Delta b_i, \\ c_{i+1} &= c_i + \Delta c_i. \end{aligned} \tag{28}$$

Using relations (28) in the quadratic equation at time $i+1$, one can write it as given by equation (29):

$$(a_i + \Delta a_i)(x_1^i + \Delta_i)^2 + (b_i + \Delta b_i)(x_1^i + \Delta_i) + (c_i + \Delta c_i) = 0 \tag{29}$$

Here it is assumed that x_1^i is one of the two roots of the equation at time i and this root differs from the root of the equation at time $i+1$ is given by Δ_i. Equation (29) after some simplification can be expressed as a quadratic equation in an unknown Δ_i as shown in (30).

$$a_{i+1}\Delta_i^2 + (2x_1^i a_{i+1} + b_{i+1})\Delta_i + (\Delta a_i (x_1^i)^2 + \Delta b_i (x_1^i) + \Delta c_i) = 0 \tag{30}$$

From the quadratic relation (30), one can find the value of the unknown Δ_i. This required value is given by:

$$\Delta_i^{1,2} = \frac{-(2x_1^i a_{i+1} + b_{i+1}) \pm \sqrt{(2x_1^i a_{i+1} + b_{i+1})^2 - 4a_{i+1}(\Delta a_i (x_1^i)^2 + \Delta b_i (x_1^i) + \Delta c_i)}}{2a_{i+1}} \tag{31}$$

Here $\Delta_i^{1,2}$ are two values of the roots of equation (31).

Unfortunately, relation (31) when used for finding roots of the quadratic equation for the interval $i+1$ may not be very effective in terms of computational efforts, may even be demanding more computational effort. However, it is a function of change that will determine the complexity of (31).

Table 5 below does give a few computational results. Further investigations are necessary, for better understanding of information recycling methods.

Interval	Quadratic equation	Roots by direct approach	Roots using (31)
i	$x_i^2 - 13x_i + 42 = 0$	7 and 6	-
$i+1$ **Case 1**	$x_{i+1}^2 - 10x_{i+1} + 25 = 0$ $\Delta a_i = 0, \Delta b_i = 3, \Delta c_i = -17$ $a_{i+1} = 1, b_{i+1} = -10$	5 and 5	Start with root 7, $\Delta_i^{1,2}$=-2, thus $7 \to 5$. Similarly the root 6 give $\Delta_i^{1,2}$**=-1,** $6 \to 5$.
$i+1$ **Case 2**	$x_{i+1}^2 - 10x_{i+1} + 24 = 0$ $\Delta a_i = 0, \Delta b_i = 3, \Delta c_i = -16$ $a_{i+1} = 1, b_{i+1} = -10$	6 and 4	Start with root 7, $\Delta_i^{1,2}$=-3 and -1, thus $7 \to 4$. Similarly the other value –1 results $7 \to 6$.
$i+1$ **Case 3**	$x_{i+1}^2 - 8x_{i+1} + 25 = 0$ $\Delta a_i = 0, \Delta b_i = 5, \Delta c_i = -17$ $a_{i+1} = 1, b_{i+1} = -8$	$4 \pm 3i$	Start with root 7, $\Delta_i^{1,2} = -3 \pm 3i$. Thus two roots are: $4 \pm 3i$.

Table 5: Summary of the results for the protean quadratic equations.

8. INFORMATION RECYCLING APPROACHES: SOME REMARKS

It may be noted that post optimal analysis is an information recycling approach but methods to be discovered under the category of information recycling are not necessarily going to be post optimal analysis. The idea of information recycling methods can be traced in many existing methods described in the literature, for example, time series analysis [5, 8], recurrence equations used in dynamic programming and related control processes [1, 2, 3]. Furthermore many earlier approaches [11, 12] can be classified as information recycling methods. It is the belief of the author that the proposed approach would generate more interest for further research and modify many of the existing methods for meaningful industrial applications.

For further research, one has to re-examine various mathematical methods and study their behaviour in a protean environment.

For a non-linear case, the challenge is going to be greater since non-linear functions are much more sensitive to changes than linear functions. This is evident from section 7 dealing with a quadratic equation.

Solving a problem is similar to the art of winning a war. This paper is concluded with two teachings of Master Sun taken from the book, "The Art of War" by Sun Tzu.

1. Page 59 – "Therefore those who are not thoroughly aware of use of arms cannot be thoroughly aware of the advantages of the use of arms."

2. Page 66 – " The general use for military is that it is better to keep a nation intact than to destroy it. It is better to keep an army intact than to destroy it, better to keep division intact than to destroy it, better to keep battalion intact than to destroy it, better to keep a unit intact than to destroy it."

In the above the most important message is that resources must be preserved in what ever quantity their preservation is possible, similarly in a problem solving situation calculations must be conserved in whatever quantity that may be possible.

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